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ABSTRACT

The similarity of item and ability parameter estimations was investigated using two numerical analysis techniques via marginal maximum likelihood estimation (MMLE) with a large simulated data set (n=1,000 examinees) and changing the number of quadrature points. MMLE estimation uses a numerical analysis technique to integrate examinees' abilities over the ability distribution for item and ability parameter estimations because of the difficulty of direct integration with a digital computer. For integrating ability, the values of quadrature points and the weights corresponding to each quadrature point are specified. The Gauss-Hermite quadrature formula and R. J. Mislevy's histogram solution (1984) have been used for numerical integration over the normal density function. It was determined that the Gauss-Hermite quadrature formula and Mislevy's graphical solution via MMLE estimated item and ability parameters equally when a large number of quadrature points was specified. When a small number of quadrature points was specified, Mislevy's histogram solution via the MMLE approach estimated item and ability parameters more accurately than did the Gauss-Hermite quadrature formula. Seven tables summarize the study. (Author/SLD)

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**Validity of Using Two Numerical Analysis Techniques to
Estimate Item and Ability Parameters via MMLE: Gauss-Hermite
Quadrature formula and Mislevy's Histogram Solution.**

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**Paper presented at the annual meeting of
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Validity of Using Two Numerical Analysis Techniques to Estimate Item and Ability Parameters via MMLE: Gauss-Hermite Quadrature formula and Mislevy's Histogram Solution.

Abstract

MMLE employs a numerical analysis technique to integrate examinee's abilities over the ability distribution for item and ability parameter estimations because of the difficulty of direct integration with a digital computer. For integrating ability, the values of quadrature points and the weights corresponding to each quadrature point are specified. The Gauss-Hermite quadrature formula and the Mislevy's histogram solution have been used for numerical integration over the normal density function.

This study found that the Gauss-Hermite quadrature formula and the Mislevy's graphical solution via MMLE estimated item and ability parameters equally when the large number of quadrature points were specified. When the small number of quadrature points are specified, the Mislevy's histogram solution via MMLE approach estimated item and ability parameters more accurately than the Gauss-Hermite quadrature formula.

I. Introduction

There are many procedures to estimate item and ability parameters. Birnbaum's joint estimation paradigm (1968) has been the standard procedure for the maximum likelihood estimation of item and ability parameters. Because of the incidental and structural problem in JMLE, Bock and Lieberman (1970) proposed marginal maximum likelihood estimation (MMLE) to remove the effect of incidental parameters by integrating over the ability distribution.

MMLE employs a numerical analysis technique to integrate examinee's abilities over the ability distribution because of the difficulty of direct integration with a digital computer. Under the numerical analysis theory, the problem of finding the sum of the area under the continuous curve is replaced by the simpler problem of finding the sum of the areas of a finite number of quadratures which approximate the area under the curve.

The MMLE approach and a numerical analysis theory have been implemented in the BILOG computer program (Mislevy and Bock, 1982, 1984, 1986). For integrating ability, the number of quadrature points, the values of quadrature points, and the weights corresponding to each quadrature point are specified. The Gauss-Hermite quadrature formula has been used for numerical integration over the normal density function. The table, due to Stroud and Secret (1966), for quadrature points and weights are used in the MMLE approach (Bock & Aitkin, 1981; Bock & Lieberman,

1970; Mislevy, 1984). The BILOG computer program does not actually provide the quadrature points and weights yielded by the Gauss-Hermite formula. The quadrature points and weights provided as the default values for normal density function in BILOG are based upon the graphical solution which is called Mislevy's histogram solution (Mislevy & Stocking, 1989).

Seong (in press) found that increasing the number of quadrature point improves the accuracy of estimation for item and ability parameter with a large data set when prior ability distributions were matched to underlying ability distributions. This study investigated similarity of item and ability parameter estimations of using above two numerical analysis techniques via MMLE with a large data set after changing the number of quadrature points.

II. Theory

An approximate integration is used when we are solving a functional equation for an unknown function that appears in the integrand of some integral and we are confronted with the problem of integrating experimental data. In a numerical analysis, an integral is approximated by a linear combination of the value of the integrand.

$$\int_a^b w(x)f(x)dx \approx \sum_{i=1}^n A(x_i)f(x_i). \quad (1)$$

x_1, x_2, \dots, x_n are points or abscissas usually chosen so as to

lie in the interval of integration. These are called the nodes or quadrature points. $A(x_1), A(x_2), \dots, A(x_1) \dots \dots A(x_n)$ are called weights or coefficients corresponding to quadrature points. The right side of the above equation is frequently called a rule of approximate integration, or quadrature formula or quadrature rule. A quadrature formula usually occurs in families depending upon a parameter such as the spacing between the integrand quadrature points or the number of the these points. With arbitrary nodes, formula (1) will be exact for all polynomial of degree $\leq n-1$.

The great mathematician Karl Friedrich Gauss (1866) discovered that the accuracy of the numerical integration process could be greatly increased for the special case $w(x) \equiv 1$ by a special placement of the nodes. The quadrature formulas that arise through application of his theorem are called Gaussian quadrature formulas.

Gauss-Hermite Quadrature Formula

There is a large family of numerical integrations that confirm to following pattern of the formula (1). For using the above formula, it is only necessary to know the "quadrature points" $x_1, x_2, \dots, x_1, \dots, x_n$ and "weights" $A(x_1), A(x_2), \dots, A(x_1), \dots, A(x_n)$. Gauss's remarkable results is given below (see Cheney & Kincaid, 1985 pp 192 for details):

Let q be a polynomial of degree n such that

$$\int_a^b q(x)x^i dx = 0 \quad i=0,1,\dots,n-1 \quad (2)$$

Let x_1, x_2, \dots, x_n be the roots of q . Then the formula (1) with these x_i 's as nodes will be exact for all polynomials of degree $\leq 2n-1$. With Gaussian nodes, the formula will be exact for all polynomials of degree $\leq 2n-1$. The quadrature formulas that arise as applications of this theorem are called Gaussian quadrature formula. There is a different formula for each interval (a, b) and each of n . There is also more general Gaussian formula to give appropriate values of integrals such as

$$\int_0^{\infty} f(x)e^{-x} dx, \int_a^1 f(x)(1-x^2)^{\frac{1}{2}} dx, \int_{-\infty}^{\infty} f(x)e^{-x^2} dx, \text{ etc.}$$

Over the doubly infinite interval $(-\infty < x < \infty)$, a frequently used weighing function is of the form

$$w(x) = e^{-\alpha^2 x^2} \quad (3)$$

Quadrature formula with above function is

$$\int_{-\infty}^{\infty} w(x)f(x)dx = \int_{-\infty}^{\infty} e^{-\alpha^2 x^2} f(x)dx \approx \sum_{i=1}^n H_i f(x_i) + E. \quad (4)$$

Hermite provided the relevant orthogonal polynomial equation to compute the nodes when $\alpha = 1$ or $\alpha = \frac{1}{2}$ (see Hildebrand, 1956, pp 277 for details). Hermite's orthogonal polynomial equation is given below:

$$H_n(x_i) = (-1)^n e^{x_i^2} \frac{\partial^n}{\partial x^n} (e^{-x_i^2}) = 0 \quad (5)$$

where n is the number of nodes (quadrature points).

The first four of the polynomials defined by Equation (5) calculate the values of the nodes as shown below:

$$H_0(x)=1$$

$$H_1(x_1)=2x$$

$$x_1 = 0,$$

$$H_2(x_1)=4x^2-2$$

$$x_1 = \pm 1/\sqrt{2},$$

$$H_3(x_1)=8x^3-12x$$

$$x = \pm \sqrt{3}/2 \text{ and } 0.$$

These values are given in the Stroud and Secrest's (1966) Table 5. Hermite's equation to compute weight corresponding each node is (see Hildebrand, 1956, pp 277 for details):

$$H_n = \frac{2^{n+1} n! \sqrt{\pi}}{[H_{n+1}(x_1)]^2}. \quad (6)$$

When there are two nodes, the equation for computing the weights is

$$H_2 = \frac{2^3 2! \sqrt{\pi}}{[H_3(x_1)]^2}.$$

The weights corresponding two nodes are $\pm \sqrt{\pi}/2$. When there are three nodes, the weights are $\pm \sqrt{\pi}/6$ and $2\sqrt{\pi}/3$. These values of weights are given in the Stroud and Secrest's Table 5 (1966).

From the Equation (4),

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx \approx \sum_{i=1}^n H_i f(x_i) + E.$$

The quadrature formula of a normal function is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} g(x) dx \approx \sum_{i=1}^n \bar{H}_i g(\bar{x}_i) + E. \quad (7)$$

Let $x = \sqrt{2} y$, $dx = \sqrt{2} dy$.

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} g(x) dx &= \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2} g(\sqrt{2} y) dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} g(\sqrt{2} y) dy. \end{aligned} \quad (8)$$

Let $\sqrt{2} y = j(y)$ and $h = g \cdot j$,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} g(x) dx &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} g[j(y)] dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} h(y) dy \\ &= \frac{1}{\sqrt{\pi}} \sum_{i=1}^n H_i h(x_i) \\ &= \frac{1}{\sqrt{\pi}} \sum_{i=1}^n H_i g(\sqrt{2} x_i) \\ &= \sum_{i=1}^n \frac{1}{\sqrt{\pi}} H_i g(\sqrt{2} x_i) \\ &= \sum_{i=1}^n \bar{H}_i g(\bar{x}_i). \end{aligned} \quad (9)$$

From the above equation, $\bar{H}_i = \frac{1}{\sqrt{\pi}} H_i$, and $\bar{x}_i = \sqrt{2} x_i$.

The nodes of the normal function are obtained by multiplying the Stroud and Secrest's Table 5 values by $\sqrt{2}$. The weights of the normal function are obtained by dividing the Stroud and Secrest's Table 5 values by $\sqrt{\pi}$.

Mislevy's histogram solution

Under a normal distribution, the ability scale is restricted from -4 to +4 and the ability interval is divided into $(n-1)$ numbers of rectangles having equal ability space. n indicated a specified number of quadrature points. It produces the specified number of quadrature points.

The normal probability density functions at each X_i , $f(x_i)$, are computed and then multiplied by the equal space, which is the width of the rectangle. These are values of weights, $A(X_i)$. The sum of these weights values is 1. When the sum of the weights is not exactly equal to 1 because of rounding error. The weights are divided by the sum of the weights to make that sum of the weights equal to 1. The default values of quadrature and weights satisfy three conditions for numerical integration of the normal distribution which are $\sum A(X_i) = 1$, $\sum X_i A(X_i) = 0$, and $\sum X_i^2 A(X_i) = 1$.

III. Method

Data

Simulated item response vectors (1,0) for 1,000 examinees were generated by the GENIRV computer program (Baker, 1978). The program requires an user to specify the number of items in a test, the values of item parameters, the sample size, and the type of underlying ability distribution. A test of 45 items was used in the present study because this number of items was large enough to yield stable results. Item parameters of the 45-items

for generating item response vectors under the two-parameter normal ogive ICC model were randomly chosen from within each cell of Table 1. Thirty one ability groups were used for the normally distributed 1,000 examinee data because these groupings and numbers of examinees yielded a good match to the normal ability distributions.

[Insert Table 1 about here]

Item and Ability Parameter Estimation

The generated data set was analyzed via the micro computer version of the BILOG (PC-BILOG version 1.1) program (Mislevy & Bock, 1986) specifying the number of the quadrature points, the values, and the weights. Most of the program default values were used except the number of quadrature points, the values of the quadrature points, and weights corresponding to each quadrature point by the Gauss-Hermite quadrature formula and the Mislevy's histogram solution. The number of the quadrature points for this study were 10, 20, 30, and 40.

For ability estimation, the Bayesian expected a posterior estimation (EAP) was chosen because no other estimator has smaller mean square error over the population for which the distribution of ability is specified by the prior (Bock & Mislevy, 1982; Mislevy & Stocking, 1989).

Similarity Measures

Two statistics were used to measure a similarity of item and ability parameter estimations of using two numerical analysis techniques. One is the correlation between the parameters and

the estimates of using the Gauss-Hermite quadrature formula and the Mislevy's graphical solution. Another is the absolute difference between the parameters and the estimates of using two numerical analysis techniques.

IV. Results and Conclusions

Item Discrimination Estimates

The correlations and the average absolute difference for item discrimination estimate were reported in Table 2 and Table 3.

[Insert Table 2 and Table 3 about here]

The correlations between the item discrimination parameters and the estimates of using the Mislevy's graphical solution were almost equal to those between the item discrimination parameters and the estimates by using the Gauss-Hermite quadrature formula. The average absolute difference of the item discrimination estimates by using the Mislevy's histogram solution were smaller than those by using the Gauss-Hermite quadrature formula when the number of quadrature points were small.

Item Difficulty Estimates

The correlations and the average absolute difference for item difficulty estimate were reported in Table 4 and Table 5.

[Insert Table 4 and Table 5 about here]

The correlations between the item difficulty parameters and the estimates by using the Mislevy's graphical solution were exactly equal to those between the item difficulty parameters and the estimates by using the Gauss-Hermite quadrature formula.

When the number of quadrature points was ten, the average absolute difference for the item difficulty estimates of using the Mislevy's histogram solution was smaller than that of using the Gauss-Hermite quadrature formula.

Ability Estimates

The correlations and the average absolute difference for ability estimate were reported in Table 6 and Table 7.

[Insert Table 6 and Table 7 about here]

When ten quadrature points were chosen, the correlation between the ability parameters and the estimates of using the Mislevy's histogram solution was larger than that of using the Gauss-Hermite quadrature formula. The average absolute difference for ability estimate of using the Mislevy's histogram solution were smaller than those of using the Gauss-Hermite quadrature formula when the number of quadrature points were ten and twenty.

When the small number of quadrature points are specified, the Mislevy's histogram solution via MMLE approach estimates item and ability parameters more accurately than the Gauss-Hermite quadrature formula. The choice of numerical analysis technique may not be an important consideration for item and ability parameter estimations via MMLE when the large number of quadrature points are chosen. When estimating item and ability parameters via MMLE with small number of quadrature points, this study suggests using the Mislevy's histogram solution rather than the Gauss-Hermite quadrature formula.

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Table 1
Item Parameters of 45 Items

	Item difficulty (θ)			Total
	-1.00	0.00	+1.00	
Item discr1. (α)				
.3 - .5	5	5	5	15
.6 - .8	5	5	5	15
.9 -1.1	5	5	5	15
Total	15	15	15	45.

Table 2

Correlations between the Item Discrimination Parameter and the Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.980	.981
	20	.980	.981
	30	.981	.981
	40	.981	.981

Table 3

Average Absolute difference for the Item Discrimination Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.067	.051
	20	.039	.038
	30	.038	.038
	40	.038	.038

Table 4

Correlations between the Item Difficulty Parameters and the Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.992	.992
	20	.992	.992
	30	.992	.992
	40	.992	.992

Table 5

Average Absolute difference for the Item Difficulty Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.101	.086
	20	.078	.078
	30	.078	.078
	40	.078	.078

Table 6

Correlations between the Item Ability Parameters and the Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.951	.953
	20	.957	.957
	30	.957	.957
	40	.957	.957

Table 7

Average Absolute difference for the Ability Estimates of Using the Gauss-Hermite Quadrature Formula and the Mislevy's Histogram Solution.

		Numerical Analysis Technique	
		Gauss-Hermite	Mislevy's
Number of Quadrature Points	10	.260	.248
	20	.232	.230
	30	.230	.230
	40	.230	.230